GUIDELINES AND PRACTICE PROBLEMS FOR EXAM 2

Exam 2 will cover all material presented in class since Exam 1 up to and including whatever we covered on Thursday April 10. Questions on the exam will be of the following types: Stating definitions, propositions or theorems; short answer; true-false; and presentation of a proof of a theorem. I will try to keep time-consuming calculations to a minimum. You may have to do calculations for 2×2 matrices, but for questions involving larger matrices, you may be asked to sent things up or describe the process you would use to solve the problem.

Any definitions, propositions theorems, corollaries that you need to know how to state appear in the Daily Update, and all such are candidates for questions. You will need to be able to answer brief questions about these results as well as true-false statements about these results. Most of the definitions you need to know are also in the Daily Update, but it is best to check your notes for all definitions we have given by February 20.

You will also be responsible for working any type of problem that was previously assigned as homework.

On the Exam you will be required to state and provide a proof of one of the following Theorems.

(i) The spectral theorem for 2×2 symmetric matrices.

(ii) Statement and proof of the Singular Value Theorem, in the case that the matrix A is 2×2. (For this, just adapt the steps given in class with the justifications of the steps.)

(iii) Statement and proof of the Graham-Schmidt process for a real inner product space, applied to the linearly independent set of vectors $\{w_1, w_2, w_2\}$.

Practice Problems

1. Find the singular value decomposition for $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$. Then find the pseudo-inverse of A.

2. Find orthogonal matrices that diagonalize: $A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$.

3. For the following matrices, find the characteristic polynomial, the eigenvalues, the dimensions of each eigenspace and determine whether or not the matrices are diagonalizable:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$$

4. Consider $\mathbb{R}^4$ with the usual dot product as its inner product. Let W be the subspace of $\mathbb{R}^4$ with basis

$$v_1 := \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) Use the Gram-Schmidt process to find an orthogonal basis for W; (ii) Then find an orthonormal basis for W; (iii) Find an orthonormal basis for $W^\perp$.

5. Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvalues for $B = \begin{pmatrix} 2 & 4 & 4 \\ 4 & 4 & 8 \\ 4 & 4 & 8 \end{pmatrix}$.

6. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$. For v, w column vectors in $\mathbb{R}^2$, define $\langle v, w \rangle := v^t A w$, where the product $v^t A w$ is just matrix multiplication. This gives a new inner product on $\mathbb{R}^2$. Find an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product. For $v := \begin{pmatrix} a \\ b \end{pmatrix}$, verify that $\langle v, v \rangle \geq 0$ and equals zero exactly when $v = \vec{0}$.

7. Show that the matrix $A = \begin{pmatrix} 1 & 1+i \\ 1+i & 1 \end{pmatrix}$ is normal, but not self-adjoint. Then find a unitary matrix that diagonalizes A.

Math 590 Sol'ns to Practice Problems for Exam 2

(1) $A^tA = \begin{pmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{pmatrix}\begin{pmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{pmatrix} = \begin{pmatrix}17 & 8\\ 8 & 17\end{pmatrix}$

$p_{A^tA}(x) = (x-17)^2 - 64 = (x-25)(x-9) \Rightarrow \lambda_1 = 25, \lambda_2 = 9$

$\sigma_1 = \sqrt{25} = 5, \quad \sigma_2 = \sqrt{9} = 3.$ Set $\Sigma = \begin{pmatrix}5 & 0\\ 0 & 3\\ 0 & 0\end{pmatrix}$

E_{25} = null space of $\begin{pmatrix}-8 & 8\\ 8 & -8\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} \Rightarrow \begin{pmatrix}1/\sqrt{2}\\ +1/\sqrt{2}\end{pmatrix} = u_1$

E_9 = null space of $\begin{pmatrix}8 & 8\\ 8 & 8\end{pmatrix} \rightarrow \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} \Rightarrow \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} = u_2$

Set $P = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$,

$v_1 = \frac{1}{5} A u_1 = 1/5 \begin{pmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}5/\sqrt{2}\\ 5/\sqrt{2}\\ 0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{pmatrix}$

$v_2 = \frac{1}{3} A u_2 = \frac{1}{3}\begin{pmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\\ 4/\sqrt{2}\end{pmatrix}$.

Find v_3' orthogonal to v_1, v_2: Take $v_3' = \begin{pmatrix}1\\ -1\\ -1/2\end{pmatrix}$

and make it have unit length: $v_3 = \frac{2}{\sqrt{5}}\begin{pmatrix}+1\\ -1\\ -1/2\end{pmatrix}$

Set $Q = \begin{pmatrix}1/\sqrt{2} & 1/3\sqrt{2} & 2/\sqrt{5}\\ 1/\sqrt{2} & -1/3\sqrt{2} & -2/\sqrt{5}\\ 0 & 4/3\sqrt{2} & -1/\sqrt{5}\end{pmatrix}$.

Then $A = Q\Sigma P^t$

② $A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \Rightarrow P_A(x) = (x-3)^2 - 4 = x^2 - 6x + 5$

$= (x-5)(x-1)$

$E_5 =$ null space of $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow u_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$E_1 = \underset{\text{null space}}{} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow u_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

orthogonal matrix

Set $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \Rightarrow Q^t = Q^{-1} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$\Rightarrow Q^{-1} A Q = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$

③ $P_A(x) = (x-3)(x-2)(x-1)$ has distinct eigenvalues $\Rightarrow$ A is diagonalizable

so each eigenspace is one dim'l

$P_B(x) = (x-1)^2(x-4)^2$.

$E_{-1} =$ null space of $\begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ has rank 3 $\Rightarrow$ dim null space $= 1$

$\therefore$ geom. multiplicity < alg. multiplicity $\Rightarrow$ <u>NOT</u> diagonalizable. Same for E_4

$P_c(x) = (x-1)^2(x+3) \Rightarrow \lambda = -3, 1$ are eigenvalues

$$E_{-3} = \text{null space } \begin{pmatrix} 8 & -4 & 4 \\ 12 & -8 & 12 \\ 4 & -4 & 8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow v = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ is basis for $E_{-3} \Rightarrow \dim E_{-3} < 2 \Rightarrow C$ is not diagonalizable. However, E_1 has dimension $= 2$.

(4) Take $w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$. Take $w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1/2}{3/2}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/6 \\ -1/6 \\ 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}$$

$\therefore u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \frac{\sqrt{2}}{\sqrt{3}}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$, $u_3 = \frac{\sqrt{3}}{2}\begin{pmatrix} 1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}$

is an ONB for W

To find an ONB for $W^\perp$ it suffices to extend u_1, u_2, u_3 to an ONB for $\mathbb{R}^4$.

Note that $v_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$ is orthogonal to u_1, u_2, u_3,

So we take $u_4 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$.

5. $p_B(x) = x^2(x-18) \Rightarrow x = 0, 18$ are Evalues

$E_0 = \text{nullspace} \begin{pmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

is a basis for E_0. We need ONB for E_0.

Set $w_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \frac{-1}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -2/5 \\ -4/5 \\ 1 \end{pmatrix}$. Take $u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \frac{5}{3\sqrt{5}}\begin{pmatrix} -2/5 \\ -4/5 \\ 1 \end{pmatrix}$

$E_{18} = \text{nullspace}$

$\to \begin{pmatrix} -16 & 4 & 4 \\ 4 & -10 & 8 \\ 4 & 8 & -10 \end{pmatrix} \to \begin{pmatrix} 4 & -10 & 8 \\ -16 & 4 & 4 \\ 4 & 8 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & -5/2 & 2 \\ 0 & -36 & 36 \\ 0 & 18 & -18 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -5/2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to v = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$ is basis $\Rightarrow u_3 = \frac{2}{3}\begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$.

$\therefore Q = [u_1\ u_2\ u_3]$ is orthogonal and diagonalizes B.

6. $\langle v, v\rangle = (a \ \ b)\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = (a+b \quad a+3b)\begin{pmatrix}a\\ b\end{pmatrix}$

$= a^2 + ab + ab + 3b^2 = a^2 + 2ab + 3b^2$

$= (a+b)^2 + b^2 \geq 0.$ If $= 0 \Rightarrow b = 0 \Rightarrow a^2 = 0$
$\Rightarrow a = 0$

$\Rightarrow v = 0.$

To find ONB, start with $v_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$ a basis.

Take $w_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$, $w_2 = v_2 - \dfrac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1.$

$\langle v_2, w_1\rangle = (0 \ \ 1)\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = (1 \ \ 3)\begin{pmatrix}1\\ 0\end{pmatrix} = 1$

$\langle w_1, w_1\rangle = (1 \ \ 0)\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = (1 \ \ 1)\begin{pmatrix}1\\ 0\end{pmatrix} = 1$

$\Rightarrow w_2 = \begin{pmatrix}0\\ 1\end{pmatrix} - \frac{1}{1} \cdot \begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}-1\\ 1\end{pmatrix}.$

$\langle w_1, w_1\rangle = 1 \Rightarrow u_1 = w_1 = \begin{pmatrix}1\\ 0\end{pmatrix}.$

$\langle w_2, w_2\rangle = (-1 \ \ 1)\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = (0 \ \ 2)\begin{pmatrix}-1\\ 1\end{pmatrix} = 2$

$\Rightarrow \|w_2\| = \sqrt{2} \Rightarrow u_2 = \begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$

$\therefore \begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} =$ ONB wrt $\langle\, ,\rangle.$

7. $AA^* = \begin{pmatrix} 1 & 1+i \\ 1+i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-i \\ 1-i & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$A^*A = \begin{pmatrix} 1 & 1-i \\ 1-i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1+i \\ 1+i & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \Rightarrow A$ is normal.

$P_A(x) = \begin{vmatrix} x-1 & -1-i \\ -1-i & x-1 \end{vmatrix} = (x-1)^2 - (1+i)^2$
$= x^2 - 2x + 1 - (1+2i-1)$

$= x^2 - 2x + 1 - 2i = (x+i)(x-2-i) \Rightarrow \lambda = -i,\ 2+i$

E_{-i} = null space $\begin{pmatrix} 1+i & 1+i \\ 1+i & 1+i \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ = unit basis elt

E_{2+i} = null space $\begin{pmatrix} -1-i & 1+i \\ 1+i & -1-i \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ = unit basis elt

$\therefore Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ diagonalizes A.

Note $A^* = \begin{pmatrix} 1 & 1-i \\ 1-i & 1 \end{pmatrix} \neq A$, so A is not self adjoint